Theoretical mechanics. Discontinues systems.

Vibroimpact processes in systems with the large number impact pairs and distributed impact elements

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The dynamic effects are considered and described, caused by different wavy conditions in discrete and distributed vibroimpact systems. We examine such systems in some cases when we have multiple collisions of subsystems or when the sizes of objects are those that it is necessary to consider wave processes occurring in colliding pairs. Such as, for example, different mechanical objects, which elements suffer impacts; various extended constructions, vibrating near pointwise, gridwork and continuous movement obstacles; vibroconductive and vibroarresting constructions, equipped with facilities with solitary and multiple breaks, etc. Original calculation methods are presented, based on the frequency-time analysis and other methods of the modern nonlinear mechanics. Multiple nonlinear effects are theoretically described, which are related to the formation of specific nonlinear waves with trapezoid profiles; advent of localizations of intensive impacts in some areas of constructions; generation of higher harmonic components; generation of non-synchronic and chaotic movements, etc.. This paper is a trial of a brief review.

1 Systems with parallel impact pairs

1.1 Basic models. Systems with parallel impact pairs include complex vibroimpact systems, in which some of the elements of one (basic) subsystem constitute impact pairs with elements of other subsystems, and each element of the basic subsystem can be incorporated in only one impact pair. Examples of such systems are shown in Fig.1

Fig.1,a shows a system with basic subsystem presented by a string with $N$ balls fixed on it. In this case the balls collide with a rigid one-side restrictions, which constitute the second subsystem. Naturally, the restriction may also be double-sided.

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Besides that, the balls may constitute impact pairs with elements of more complicated form. Fig.1.b displays a transversally oscillating string or beam interacting with pointwise restrictions. Here the subsystem with distributed parameters acts as a basic.

**1.2 Motion equations.** In general case for one-dimensional system with $N$ parallel impact pairs (Fig.1,a) we can write motion equation as

$$\rho u_{tt}C^u[u]+B^u[u]=\sum_{j=1}^{N}\{-m_ku_k(x,t)+P_k(t)\}-\Phi[u,u_t]\delta(x-x_j).$$

where: $u(x,t)$ is the desired displacement; $C^u$ and $B^u$ are linear "elastic" and "dissipative" operators of the system; $\rho$ is linear density of the material, $m_k$ are masses of the bodies located in points $x_k$; T-periodic driving forces $P_k(t)$ act, for instance, in points of location of impact pairs $x_k$; $\delta(t)$ is Dirac’s delta-function, and $\Phi[u,u_t]$ is the force of impact interaction, defined by relations following from Newton’s or other hypothesis and assumptions concerning the nature of the process. For example, we can write here [1], $\Phi[u,u_t]\delta(x-x_j)$

$$=J_k\delta^T(t-t_j),$$

where $\delta^T(t)$ is T-periodic delta-function; $J_k$ - the impact impulse in the $k$-th pair; and $t_j$ is the phase of impact. We do not have possibility to consider formal features of the notation for a force of impact interaction. So we recommend, in particular, works [1-9].

Independent of the system structure the motion equations (1), and other may be written uniformly in the operator form. For the required movements field $u(x,t)$:

$$u(x,t)=\sum_{j=1}^{N}L(x, x_j; p)[P_f(t)-\Phi[u(x,t); pu(x,t)]\delta(x-x_j)],$$

where the dynamic compliance operators $L(x,y;p)$ are determined by the structures of the interacting subsystems; $p=d/dt$. For description of the system with lumped parameters in the operator equation (2) we should set $x=x_j$.

**1.3 Analysis methods.** The study of systems with parallel impact pairs is carried out numerically or by means of frequency-temporal analysis methods [1,6]. In case of a periodic outside excitation in order to find the $T$-periodic modes we can obtain the integral equation of periodic motion [6]:

![Diagram](image-url)
\[ u(x,t) = u_0(x,t) + \sum_{j=1}^{N} \int_{0}^{T} \chi(x,x_j;t-s) \Phi(u_j,u_{jt}) ds \]  

(3)

Here \( u_0(x,t) \) is stationary movement field under driving forces in the absence of collidings; \( u_j = u(x_j,t) \), and also

\[ \chi(x,x_j;t-s) = T \sum_{k=-\infty}^{\infty} L(x,x_j;ik\omega) \exp(ik\omega t) \]  

(4)

is periodic Green function (PGF), compliant to the operator \( L(x,x_j;p) \) ([1]). If assumed that the impact is momentary and acts one time during the motion period, so that the function \( \Phi(u_j,u_{jt}) \) can be presented as a combination of periodic delta-functions. The equation (3) can be reduced to the following representation of the vibroimpact process:

\[ u(x,t) = u_0(x,t) + \sum_{j=1}^{N} J_j \omega(x,x_j;t-t_j) \]  

(5)

where \( J_j \) is impact forces impulse in the \( j \)-th impact pair; \( t_j \) is moment of the impact in this pair.

We shall call the representation (5) "2N-parametric". The unknown motion parameters can be sought from the impact conditions

\[ u(x,t_j) = ?_j, \quad J_j = m_j(R_j+1)u_t(x_j,t_j-0). \]  

(6)

Here \( ?_j, R_j \) - clearance and restitution ratio in the \( j \)-th impact pair.

The solutions obtained should be analysed regarding the stability and feasibility of geometric conditions of the type \( u(x,t_j) \geq ?_j \). The final solution of the problem in a visible analytical general form can be obtained for a limited number of models of such kind, however, for particular parameters it is almost always possible to find a corresponding numerical-analytical solution. Besides that, basing on the representation (5), (6) it is possible to build up some approximate solutions.

1.4 The essential dynamic effects. Further we will discuss some effects revealed as a result of analyzing the model Fig.1,a with a periodic structure: for each \( j \) all \( m_j, b_j \) and \( c_j \) values are the same. The outside excitation was chosen sinusoidal.

The main result (see also clause 2.5 and [5, 6, 10]) is the discovery of existence of periodic modes with synchronous impacts in remote impact pairs. Such modes are calls "claps". At excitation of these modes the string with the balls fixed on it generates one-, two- or multitrapezoid forms quite similar to the corresponding inherent oscillations forms of a linear system in terms of alternation of nodes and crests of waves, and feasible in frequency areas situated at the right from the inherent frequencies of the corresponding linear system. Fig.1,c,d shows an example of claps for two lowest forms in the system with six symmetrical impact pairs.
The trapezoid form appears even with two impact elements. This is not surprising, since in this case exactly this form is the first inherent form of linear system oscillations. The same form with two elements in the crest of the wave is one of the higher inherent forms for systems containing $6n-1$ elements, $n=1,2,\ldots$ and therefore for them realization of such form in vibroimpact processes is quite natural.

The surprising thing is that trapezoid form become dominating as the number of elements in oscillations of the crest of the wave increases. At realization of one- and multitrapezoid standing clap-waves the dynamic effects arise which are typical of systems with a single impact pair: phenomena of delaying by frequency and amplitude, and also hard initiation (see [2] and clause 2.4). Thus the behavior of clap-waves is in many respects similar to the behavior of a system with a single impact pair and, particularly, of a simple "impact oscillator".

Fig.3 displays the amplitude-and-frequency characteristic of a system with parallel symmetrically situated obstacles in a neighborhood of two lower forms of the linear system. At amplitudes which are less than the clearance ($a<\theta$) the branches of resonance curves of the linear system are being realized. After reaching of obstacles a vibroimpact process of the clap type takes place, number of bodies involved in the clap depending on the inducing forces level. The figure illustrates the above mentioned analogy between dynamics of claps and vibroimpact systems with a single impact pair.

Besides the above mentioned, these "similar" effects include, in particular, appearance of ambiguity areas of amplitude-and-frequency characteristics and breaking of oscillations away from one stable branch of resonance curve to another (arrows in Fig.2).

It should be noted that distributed systems with multiple parallel impact pairs like shown in Fig.1,b have similar characteristics.

The numerical analysis made it possible to find in these systems modes of significantly more complicated nature, which can be considered as chaotic.

It would be necessary specially to notice, that in some cases in this systems we can detect the modes of motion of a solitons type [5, 10, 11].
1.5 Remark: systems with serial impact pairs. Systems with serial impact pairs include vibro-impact systems with large numbers of degrees of freedom in which all elements except for outer ones are involved in two or more impact pairs. Many of the typical models of such systems are shown in [12,13].

2 Systems with distributed impact elements.

2.1 About these systems. Now we’ll briefly examine the problems concerned with the models of systems with distributed impact elements, which can appear while studying discrete systems of the type (1) as the corresponding long-wave approximations, or while using axiomatic approaches.

It is reasonable to consider distributed impact elements, particularly, when studying vibroconductive (vibroinhibiting) continuous media with complicated structures, and also when studying various extended objects (strings, threads, flexible bars, cables, ropes, beams and other systems), moving near some solid or gridwork obstacles, partitions, walls, gridworks etc.

Reference [14] considered models of distributed linear media of complicated structure. The most specific feature of mentioned structures is the presence of two main “medium parts”, namely: “carrier” and so called “attached” parts. Dynamic description of such systems, in general, consists of two groups of motion equations - reflecting “carrier” and “attached” subjects constrained behavior respectively. Likewise all the models in the multipolar mechanics, the concept of a point is a subject of significant revision: its state can be defined by unspecified number of kinematic parameters.

Such models utilization arise to be productive for solution of some practical problems, such as dynamical analysis of vibrostates of complicated mechanical structures, consisting of, said, distributed single dimensioned “carrier” and of gross number flexurally “attached” solid devices.

It seems that mentioned considerations could be useful by the dynamic models creation for systems consisting of “carrier” and “attached” parts with multiple breaks in it. Due to the system nature the possibility of different type collisions arises in the “attached” subsystems.

Then we consider completely distributed model. Thus, assuming the impact pairs to be “spreaded” within a certain space, we can use the concept of distributed impact elements.

Now, let’s overview briefly some papers concerning the distributed impact elements. We can obtain the model of distributed impact element by at least two ways.

Firstly, in some cases it appears impossible to disregard wave process arising in the impact pairs itself. Impacting bodies can’t be considered as the solid bodies since the
lengths of the waves generated by collisions are comparable with the impacting surfaces dimensions.

Secondly, considering the dynamic system with amount of convenient impact pairs to be large enough we can perform long-wave approximation and transit to the distributed model with distributed impact element.

In this way, starting with an abstract model of “regularly” concentrated beds on the strained thread colliding with the solid stationary obstacle [15] we can obtain basic physical model of the string, colliding with the solid stationary obstacle (see Fig.2.c and Fig.3 - here the double-side obstacle is shown.).

The number of technical applications for such a model is huge. The mentioned above basic model, initially considered in the paper [7], is now examined deeply enough theoretically [8, 9, 15, 16] and experimentally [15-17] (further references may be found in above papers).

The idea to use the concept of distributed impact elements seems to be also productive for dynamic analysis of great number of lightly damped vibroisolated devices attached over the flexible carrier with the motion limiters and interacting dynamically via carrier (see Fig.4). This model is an exact model of strongly nonlinear vibroconductive media with complicated structure.

Evidently, for the first time this problem was considered in [19]. The problem of vibration propagation through a strongly nonlinear vibroconductor (bar with the distributed impact element, modeling the two degree of freedom attached equipment) was solved in this paper. Main definitions and solution method, based on so-called non-linear forms of oscillations (periodical motion conditions having certain symmetric properties) and generalized methods of frequency-time analysis of vibroimpact processes [1] was suggested.

2.2 Models.

2.2.1 Let’s postulate the existence of some elastic carrier medium, which motion being described by function $u(x,t) \in \mathbb{R}^3 (x \in \mathbb{R}^3, t \in \mathbb{R})$, subjected to the classical Lame equation

$$\rho u_{tt} = (\lambda + \mu) \text{grad div } u + \Delta u + F$$

(7)
where $\rho$ - density of the carrier medium and Lame factors $\lambda$, $\mu$ characterize the carrier elasticity properties.

Let the intensity of the volumetric forces has the following structure: $F = F_0 + F_1$, where $F_1$-dictated vector, and $F_0(X,t)=C_1(u - Y_k^{(1)}) - C_2(u - Y_k^{(2)})$, where it is assumed that each point of the medium carries an elemental impact pair, consisting of two interacting linear stationary subsystems $A^{(1)}(x)$ and $A^{(2)}(x)$, being described by dynamic compliance operators: $L_{ij}^{(1)}(p)=O(p^{-2})$ and $L_{ls}^{(1)}(p)=O(p^{-2})$; subscript indexes $q, j, l, s$ are varying within some sets, depending upon the dimensionalities of interacting subsystems, which parameters can be determined, in general, by the $x$ value. In order to close the system of motions equations, let’s add the following relationships:

$$Y_k^{(1,2)} = L_{nk}^{(1,2)}(p)C_{1,2}u \pm L_{kk}^{(1,2)}(p)\Phi_0(Y^0, Y^0) + f_k^{(1,2)}, \quad (8)$$
$$Y_n^{(1,2)} = L_{nn}^{(1,2)}(p)C_{1,2}u \pm L_{kn}^{(1,2)}(p)\Phi_0(Y^0, Y^0) + f_n^{(1,2)}, \quad (9)$$

where $Y_n^{(1,2)}(x,t)$ and $Y_k^{(1,2)}(x,t)$ - displacements of the hanger and contact points; $Y^0=Y_k^{(2)}-Y_k^{(1)}$ -relative convergence of impact elements of the interacting subsystems to which linear densities $m_1(x)$ and $m_2(x)$ are referred; $\Phi\{Y_n^{(1,2)}, Y_k^{(1,2)}, \partial / \partial t [Y_n^{(1,2)}], \partial / \partial t [Y_k^{(1,2)}]\}$ - density of impact force (for the system $A^{(1)}(x)$ in (8) and (9) we choose the “plus” sign and for $A^{(2)}(x)$ - “minus”); these equations can also include some functions $f$, describing an additional external effects.

The boundary conditions for (1) which models the carrier construction (namely, physical and geometrical characteristics) are the same as in the classical case. The frequency properties of the amortized equipment generating vibroimpact processes can be obtained from the model of the added medium part, containing the impact element. The connection mechanism of the carrier and added parts determines the structure of global vibration field.

This approach, adapted for analysis of the collective effect of impact pairs, possibly sacrifices information on “individual” features of certain concrete system elements, as well as effects, which can only be clarified if considering the model’s discreteness.

2.2.2 Let’s consider the second example [17]: supported flexible beam with total length $L$ (see Fig.5,a) vibrates over the solid wall. Beam parameters are denoted as following: $\rho$ - linear density, $E, G$ - elasticity modules of the first and second order, $F, \Gamma$ - cross sectional area and moment of inertia, $k'$- factor describing the stress distribution nonuniformity over the beam cross section, $P(x,t)$ - external force distribution, $\Delta$ - clearance between the beam equilibrium position and wall. Using the Timoshenko beam model and denoting by $u(x,t)$ and $\{Y(x,t)-u_e(x,t)\}$ the beam instantaneous linear and
angular deflection shape, \( \Phi(u) \) impact force distribution, the motion equations will be written as following:

\[
\rho u_{tt} - k' F G u_{x x} + k' F G Y_{2x} + \Phi(u) = P, \quad E I Y_{2x} - \rho F' Y_{2i} = 0 \quad (10)
\]

with the boundary conditions \( u(0,t) = u(L,t) = 0, \quad t \in ]-\infty, \infty[, \quad x \in [0,L] \).

**2.3 On the analysis methods.** In spite of the complexity of the motion equations, in some cases \([19,20]\) it appears possible to perform not only the corresponding numerical analysis of mentioned models, but also to obtain an approximate analytical representation of the required motion distributions as well.

These representations can be obtained by means of the modified frequency-time analysis of vibroimpact processes and other methods of the modern non-linear mechanics.

In many cases the equations of motion can be reduced to the following "two-functions" representation of the vibroimpact process \([u(x,t)]:\)

\[
u(x,t) = u_0(x,t) - \int J(z) \chi[x, z ; \Phi(t(z)) \] dz , \quad X
\]

where \( u_0(x,t) \) - steady-state movement field under inducing forces in the absence of impact interactions; \( X \) - some integration area; \( \chi(x,z,z) \) - PGF of system.

Unknown two functions \( J(x) \) (the impact force impulse distribution) and \( \Phi(x) \) (the impact force phase distribution) can be sought from the impact conditions.

**2.4 Two principal examples.**

**2.4.1 The first example** - the longitudinal waves propagation in the rod with flexurally attached in its every point impact pair (see Fig.6). Applying above mentioned approach we can obtain the following system of equations:

\[
\rho u_{tt} - E a u_{xx} + C(u - y) = 0 \\
m_1(y_{tt} + y'_{tt}) + \Phi(y') = 0 \quad (12)
\]

Here \( u, y, y' \) - displacements of system structural elements with instant coordinate \( x \); \( u = u(x,t) \) is carrier displacements; \( y = y(x,t) \), and \( y' = y''(x,t) \) is displacements of attached elements, \( \rho, E, A \) is- linear density, cross sectional area and Young modules of rod material respectively;
\( \Phi(y') \) is impact force density; \( m, m_1 \) are linear densities of attached parts. Boundary conditions for (12) looks like: \( u(0,t)=0, \ u(L,t)=\mu \cos \omega t \), where \( L \) - the length of the rod.

Conditions of Newton’s impact are as follows: \( y^0(x, \Psi(x))=\Delta, \ \Psi(x)=M(1+R)y^0(x, \Psi(x)-0)>0 \). Here \( \Psi(x) \) is impact force phase distribution, \( J(x) \) is impact force impulse distribution, \( M=mm/\left(m+m_1\right) \) is reduced mass distribution, \( R \) - restitution ratio. In general case values \( M, R \) and \( \Delta \) are varied depending on \( x \).

2.4.2 Let's consider again our second example - see clause 2.2.2. Supposing the existence of periodical single-impact motion, the impact force distribution may be considered as follows: \( \Phi(u)=J(X)\delta^\tau[t-\varphi(x)] \), where \( J(x)>0 \) impact force impulse distribution, \( \varphi(x) \) - impact phase distribution, \( T \) - period of oscillation. In order to describe the “clap” shape (see Fig.5,b.), \( \varphi(x)=\varphi=\text{const} \), the Newton impact conditions may be formulated as follows: \( u(x,0)\big|_{0}=u(x,0)\big|_{+}=u(x,0)\big|_{-}=(\Delta x_{i})x \ {\text{if}} \ x\in[0,L-x_{i}] \), and \( u(x,0)=\Delta \ {\text{if}} \ x\in[x_{i},L-x_{i}] \), and \( u(x,0)=\left(\Delta x_{i}\right)(L-x) \ {\text{if}} \ x\in[L-x_{i},L] \), and also \( u_t(x,0)=V_0 \ {\text{if}} \ x\in[x_{i},L-x_{i}] \), and \( u_t(x,0)=0 \ {\text{if}} \ x\in[x_{i},L-x_{i}] \); \( u_t(x,0)\big|_{+}=Ru_t(x,0)\big|_{-} \). Under such suppositions: \( J(x)=(1+R)\rho V_0=\text{const}; \ \varphi(X)=\text{const}=\varphi(x\in[x_{i},L-x_{i}]) \) and conditions of impact have to be considered for the beam midpoint which is obviously involved in vibroimpact process.

2.5 The essential dynamic effects. Calculations of a number of concrete systems enables to find out and systematize various dynamical effects arising there. Let's note the most specific and impressive ones:

2.5.1 Vibroconductors with impact elements:

- advent of spatial areas of the most intensive collisions;
- advent of spatial areas of "transparency" and "locking" of vibroconductive media for the main oscillation tone. The structure of these areas can be very complicated and determined by some specific resonance relationship depending upon the physical and geometric properties of the carrier part of the vibroconductor, upon the structure and frequency properties of the attached equipment, containing certain types of impact pairs, and, of course, upon the dissipative factors;
- generation of the higher harmonic components of propagating vibration, including the case of a main tone delay;
- advent of combined (particularly subharmonic) motion conditions;
- advent of chaotic motions.

2.5.2 One-dimensional extended objects (strings, beams, etc.) vibrating near straight obstacles:

- arising of trapezoid standing waves ("claps"), characterized synchronous coming of distant points of distributed systems to the restrictions;
emergence of higher forms of claps (multitrapezoid standing waves);

arising, as the claps occur, of the effects, characteristic to impact oscillators: delaying, multivaluedness of amplitude-and--frequency characteristic, feasibility of "hard excitation" and others; arising of near-periodic standing waves;

retension of trapezoid profiles of standing waves at different kinds of outside and self-running excitation;

arising of standing waves characterized by specific profiles of complex nature (quasi-adhesion, emergence of "inside out" configurations, etc.).

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